Using Steffe's Fraction

Recognizing schemes, which are different from strategies, can help teachers understand their students' thinking about fractions.

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This feature is a companion to Norton and McCloskey's "Modeling Students' Mathematics Using Steffe's Fraction Schemes," which appeared in the August 2008 issue of *Teaching Children Mathematics* (*TCM*). Readers were introduced to results from Steffe's research on early to middle elementary students' operations with fractions. The intention in that article, as with this one, is for teachers to use the discussion as a resource for eliciting, interpreting, and supporting their students' reasoning with fractions. This article follows a similar format; fraction schemes are presented, along with examples of fifth-grade and sixth-grade student work on problem-solving tasks.

Middle school teachers are well aware of the difficulties that many students have with fractions. Efforts to assess and support students' reasoning with fundamental fraction concepts can be more valuable than spending classroom time practicing the standard algorithms for computations with fractions. In fact, we have found that overemphasizing procedural fluency with fraction operations (adding, subtracting, multiplying, and dividing) circumvents the development of important understandings. We encourage teachers to take the time and effort to assess and support what their students *understand* about fractions as numbers that are produced through mental actions.

It may be surprising to some that these ways of operating are beyond the mathematical development of many high-achieving students. Readers may find, as we did when studying Steffe's research, that *they* are the ones learning new mathematics. We have tried to provide a resource for middle school teachers who may benefit from contemporary research in mathematics education. We would like to encourage interested readers to refer to our article in *TCM* (2008) for a more thorough discussion of the basic components of Steffe's fraction research (schemes and operations) as well as a presentation of the initial fraction

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Advanced Schemes

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Table 1 Fractional operations—mental actions used by students

Operation	Description	Example of Using the Operation
Unitizing	Treating an object or collection of objects as a unit or whole	Treating two hexagons as a whole (as with pattern blocks)
Partitioning	Separating a unit/whole into equal parts	Sharing a pizza among four people
Disembedding	Imaginatively pulling out a fraction from the whole while keeping the whole intact and unaltered	After the pizza has been sliced in fourths, imagining what three-fourths of the pizza would look like
Iterating	Repeating a part to produce identical copies of it	Using a one-fifth piece to identify a three-fifths piece (as with fraction rods)
Splitting	The simultaneous composition of partitioning and iterating	"This bar is five times as long as another bar. Draw the other bar."

schemes. We also recommend several other articles (Steffe 2004; Steffe and Olive 1991).

OPERATIONS VERSUS SCHEMES

Operations are mental actions that have been abstracted from experience to become available for use in various situations. In the middle grades, as well as the lower grades, there are several important fraction operations: unitizing, partitioning, disembedding, and iterating. They are summarized in **table 1** with one additional operation, splitting, which is discussed below. Operations constitute the key component of schemes; such operations, or mental actions, produce students' concepts of fractions.

Schemes are constructs used by teachers and researchers to model students' cognitive structures. We attribute schemes to students to explain and predict students' actions and in so doing provide information about instruction. It is common to misinterpret schemes as strategies because both are of interest to teachers who pay close attention to their students' problem-solving activities. The difference is that schemes describe ways of operating that usually occur outside the student's awareness. Also, schemes are activated at once, rather than in a sequential manner. We present a summary of Steffe's fraction schemes in **table 2**. The first five schemes are explored in depth in *TCM* (Norton and McCloskey 2008). This article discusses the reversible partitive fractional scheme and the iterative fractional scheme.

We caution against the temptation to teach these schemes directly to students. Their ways of operating are deeply connected to how they make meaning out of their mathematics. Attempts to circumvent or replace those ways of operating will only yield disconnected knowledge. Teachers should find ways to work within their students' ways of operating while generating a need to develop more powerful ways of operating. Although this is not easy, we hope that the terms, tasks, and student work examples will serve as a resource to teachers seeking to do just that.

The *partitive fractional scheme* is the first scheme in which students coordinate units at two levels. Given an unpartitioned piece representing four-sevenths, for example, a student working with this scheme would be able to consider the quantity in two ways. First, the student would recognize that the piece is four-sevenths of some unpartitioned whole. Second, the student understands that the piece is four iterations of the one-seventh unit while the whole is seven iterations of that same unit. In other words, the student is developing powerful ways of operating so that he or she can iterate and partition creatively to determine the size of a fractional piece.

When teachers support students by providing opportunities for multiple experiences in which students can both iterate and partition in the same setting, students may be able to abstract the splitting operation.

THE SPLITTING OPERATION

According to Steffe (2003), *splitting* is the simultaneous composition (that is, the recognition that one implies the other as an inverse operation) of partitioning and iterating. As defined in **table 1**, *partitioning* involves breaking a piece into segments; *iterating* involves duplicating a piece a number of times. Beyond partitioning and iterating sequentially, students who are "splitters" can exploit the inverse nature of these two operations to solve problems and re-create the whole from any fractional piece.

A student may be able to partition to solve the task "Find one-third of this bar" and may be able to solve the task "Find a bar 4 times as long as this bar" by iterating. However, the splitting operation would enable him or her to solve this significantly harder task: "This bar is five times as long as

Table 2 Fractional schemes—constructs used by practitioners to characterize student thinking

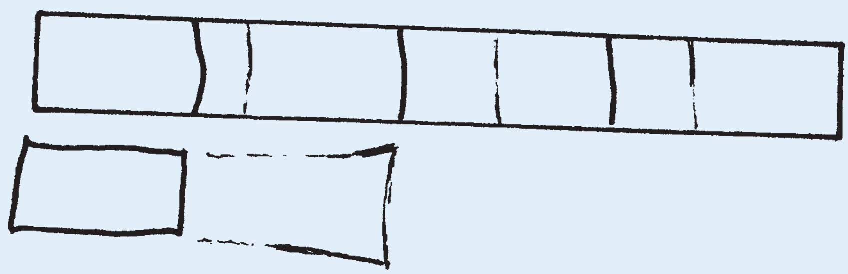
Scheme	Operations	Sample Task
Simultaneous partitioning scheme	Unitizing the whole, partitioning the continuous whole using a composite unit as a template	Share this candy bar equally among you and two friends.
Part-whole scheme	Unitizing, partitioning, disembedding a part from the partitioned whole	Show me two-thirds of the candy bar.
Equi-partitioning scheme	Unitizing, partitioning, iterating any part to determine its identity with the other parts	If you share this candy bar equally among you and two friends, show me what your piece would look like.
Partitive unit fractional scheme	Iterating a given unit fraction to produce a continuous partitioned unitized whole	If I give you this much [show a one-third piece and an unpartitioned whole], what fraction of the candy bar would you have?
Partitive fractional scheme	Unitizing, disembedding a proper fraction from the whole, hypothetically partitioning the proper fraction to produce a unit fraction, iterating the unit fraction to produce the proper fraction and the whole, coordinating unit fractions within a composite fraction (units coordinating at two levels)	If I give you this much [show an unpartitioned two-thirds piece and unpartitioned whole], what fraction of the candy bar would you have?
Reversible partitive fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a larger whole to re-create the whole	If the bar is four-fifths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.
Iterative fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a smaller whole to re-create the whole	If the bar is five-fourths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.

another bar. Draw the other bar." This latter task is posed using language that is suggestive of iterating, rather than the necessary partitioning. A student who can split recognizes the appropriate use of partitioning to arrive at the solution. This student realizes that the given piece was produced using iteration, and must, therefore, partition to re-create a single whole.

Students who can split have reached a significant stage in their understanding of fractions. Since they are able to flexibly and simultaneously iterate and partition, they are in a position to develop some advanced fraction schemes, such as the reversible partitive fractional scheme (see **table 2**). We have found that identifying students as being possible "splitters" can be helpful to teachers as they seek to foster fraction learning.

Fig. 1 This student is a "splitter." He drew the correct answer, erased it, and drew his new answer to align with the original bar. He then partitioned to verify his answer.

5. The bar shown below is four times as long as another bar. Draw the other bar.



We demonstrate with **figures 1** and **2**. In **figure 1**, the student found the correct answer, so we may conclude that he is a "splitter." Since we are interested in not merely whether students got questions right but also in what we can infer about student understanding, we cannot help but notice the additional marks the student erased.

A haphazard analysis might lead to the conclusion that this student is still tentative in his splitting. However, after our careful consideration, this student seemed to first draw a piece underneath the bar, without aligning it with the leftmost edge of the bar above. He seemed, in other words, to estimate the answer, which is not far off. This student seems to be able to use the partitioning operation imaginatively, without relying on the corresponding physical motions. If this is true and the student only later drew the partitions and disembedded one of them, then he may in fact be a very strong splitter.

The work shown in **figure 2** demonstrates other student perceptions. Although recognizing the need to partition to re-create the whole, this student confused an additive for a multiplicative relationship. He thought that *three times as* meant the same thing as *three more than.* Students make this common mistake when they encounter a multiplicative situation with only additive reasoning available.

Another common confusion with splitting tasks occurs when a student reads the phrase *three times* and iterates the given bar three times, as if it were the bar in question. Such students might benefit from tasks that ask them to identify the size of a given fraction relative to various wholes. For example, a teacher might ask:

> If this bar is the whole, what fraction is your [given] bar? What if this [new bar] bar is the whole?

Make explicit the dependence of a fraction on the whole.

REVERSIBLE PARTITIVE FRACTIONAL SCHEME

Teachers can pose the following task to elicit students' use of this scheme:

> Here is the whole. What size is your piece?

This task would best be illustrated with no partitions drawn and if the piece is some proper fraction of the whole, such as two-thirds. From here, a teacher can then assess a reversible fractional scheme, meaning that an unpartitioned fraction piece is provided. The student has to construct the *whole,* where no partitions have been drawn. A teacher can then assess the student by asking:

> This piece is three-fourths of the whole. Can you draw the whole?

The context continues to involve proper fractions, but students must reverse their previous ways of operating, which requires them to enact a splitting operation.

A quick initial assessment of **figure 3** may lead us to the conclusion that the student is using a reversible partitive fractional scheme. After all, the *other bar* that was produced looks correct. Furthermore, the final marks made are consistent with enacting a splitting operation. However, a closer look causes us to reconsider.

Notice that the given bar was originally sectioned into five pieces. A student limited to using a part-whole scheme would attempt to find 4/5 of the piece by initially partitioning the bar into fifths. This student was perturbed enough by the wording to realize that the given bar was not the whole. She may have then proceeded to partition the bar into four parts, draw an identical bar below with four parts, and add on another piece to the right end to produce the five parts in the whole. In other words, she may have solved this problem without using any iteration (much less a splitting operation), thinking only additively, within her part-whole scheme.

Alternatively, she may have used

Fig. 2 In this splitting task, the student confused an additive for a multiplicative relationship.

6. The bar shown below is three times as long as another bar. Draw the other bar.

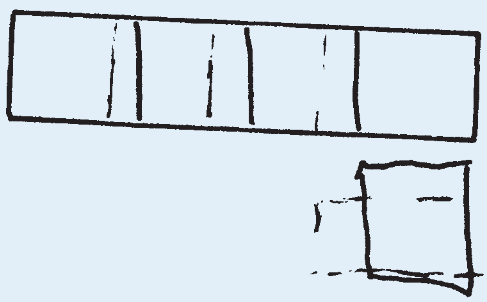
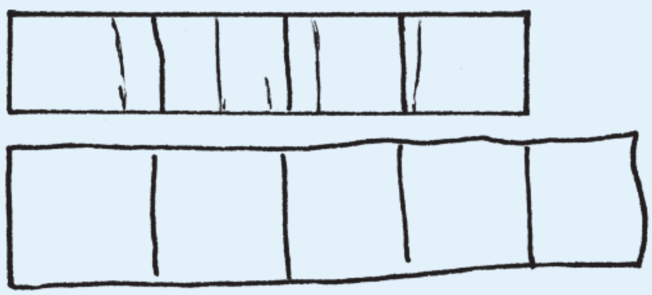


Fig. 3 This task involved a reversible partitive fractional scheme. Note that the initial partitions were in fifths and then erased.

2. The bar shown below is 4/5 as long as another bar. Draw the other bar.



iteration, eventually realizing that the one-fourth pieces in the bar should be iterated. Doing so five times would result in a pair of bars in which the top one is four-fifths of the bottom one. This reasoning is consistent with the partitive fractional scheme. However, we are hesitant to attribute even this scheme to her because we do not see indications of an iterating operation, such as small tick marks.

In either scenario, we are reasonably confident that this student is not using the reversible partitive fractional scheme that this item was designed to elicit. Remember that schemes are relatively stable. After reading the problem, a student using a reversible partitive fractional scheme would partition the given bar into fourths (physically or imaginatively), an operation that did not seem to be initially triggered for this student.

The larger lesson to be learned from this student's work is that it is impossible to guarantee a perfect correspondence between the tasks that we design and the schemes that they are intended to elicit.

The student was able to produce a correct answer on a task that was specifically and carefully designed to assess for a particular scheme. Nevertheless, she did so by relying on a less advanced scheme. This is not to say that her work is entirely discouraging. On the contrary, she shows a great deal of intelligence and creativity. As her teacher, we would find this item enlightening. We now know that this student's understanding of fractions may not be as advanced as her work on assessments may have earlier indicated. We would try to design tasks that would successfully frustrate her ways of operating, which we now know may be limited to part-whole conceptions. We are reminded, once again, of the importance of assessing the entirety of our students' work (including seemingly irrelevant markings), not simply looking for correct and incorrect final answers.

Fig. 4 The student imaginatively partitioned the top bar into fourths with this iterative fractional scheme task.

4. The bar shown below is 4/3 as long as another bar. Draw the other bar.

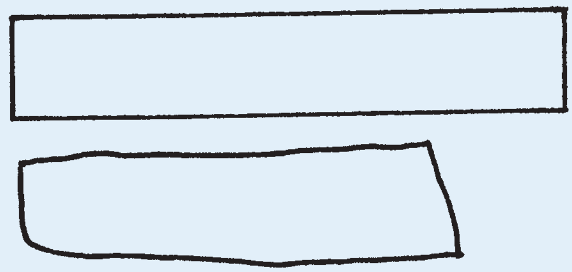
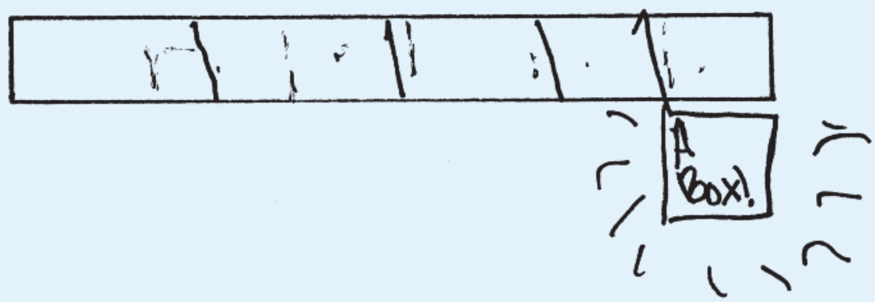


Fig. 5 Another iterative fractional scheme task. The student did not iterate a fifth of the original bar necessary four times for a whole.

2. The bar shown below is 5/4 as long as another bar. Draw the other bar.



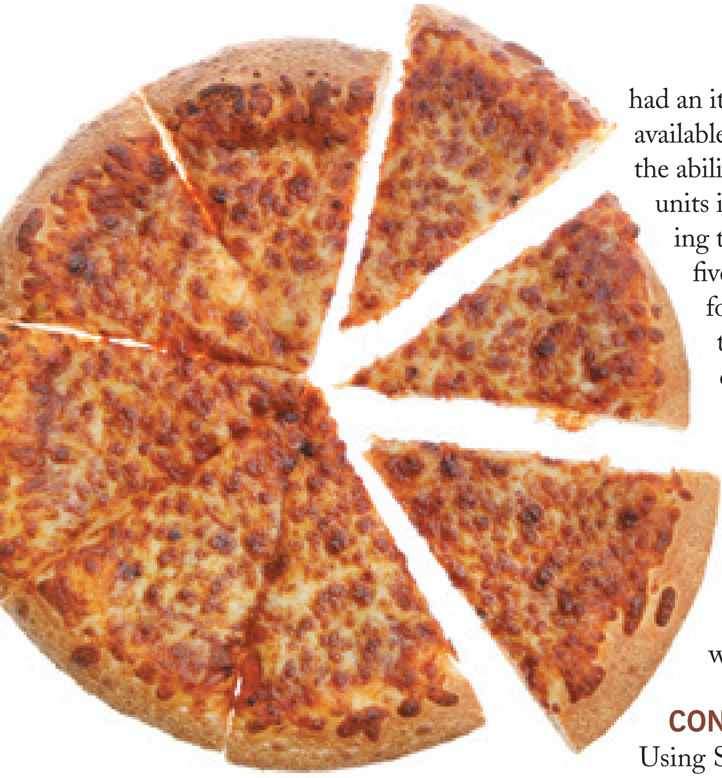
ITERATIVE FRACTIONAL SCHEME

The iterative fractional scheme, involving improper fractions, is similar to the reversible partitive fractional scheme since students use the splitting operation to partition and iterate creatively. However, the resulting answer is smaller than the given piece because of the improper fractions. An example of a task that would provide an opportunity to enact the iterative fractional scheme is this:

> This piece is four-thirds of the whole. Can you draw the whole?

The language of the task is identical to that used in the reversible partitive fractional scheme. However, the context of improper fractions constitutes a unique and significant challenge for students, as many teachers can attest. When solving such a task, the student must understand that a record of the whole is contained within the part.

The student whose work is in **figure 4** produced a correct response to this task, which was designed to elicit an iterative fractional scheme. Unfortunately, she did not leave any marks so that we could be confident about her reasoning. One hypothesis is that she imaginatively partitioned the top bar into fourths, then reasoned that the whole would be equivalent to three of those one-fourth pieces. This way of operating would be consistent with an iterative fractional scheme because she



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would understand four-thirds as being four iterations of one-third. She would anticipate that any one of those thirds could be used to reproduce the whole.

Figure 5 demonstrates that the student recognized the need to partition the bar into fifths and seemed to realize that the whole was contained within the given bar. This serves as a mild indication of a splitting operation. However, she did not iterate her fifths piece four times, as needed. This serves as a counterindication that she had an iterative fractional scheme available. She may be lacking the ability to coordinate all the units involved in reproducing the whole—each of the five sections as one-fourth, four of which reproduce the whole. It is also a bit confusing that her pieces are unequal. The marks she made at first and then erased seem to be more accurate than her final partitions, so we would want to ask her why she was unsatisfied with them.

CONCLUSION

Using Steffe's advanced fraction schemes, we describe a progression of development that upper elementary and middle school students might follow in understanding fractions. Each scheme can be viewed as a reorganization of the preceding scheme. For example, when students develop a splitting operation, they can reverse the operations of their partitive fractional scheme to produce a reversible partitive fractional scheme. Similarly, when students have developed a reversible partitive fractional scheme, they may then be ready to extend that understanding into the context of improper fractions, fostering the development of an iterative fractional scheme.

Teachers can support students' reorganizations if they understand students' current schemes and provide tasks that appropriately frustrate those ways of operating. We hope that, in addition to describing the schemes themselves, we have presented useful ideas about how to make such instructional decisions.

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Assessing Student Work

We have tried to emphasize the impossibility of assessing students' cognitive activity with certainty, but we recognize that the work of teaching requires us to make our best guesses. We encourage teachers to draw on these fraction schemes to make such inferences.

Some readers may take issue with some of our inferences. We realize that multiple valid interpretations are possible in each case. Of course, when using the fraction schemes framework to assess your students, the more data you can draw on, the better informed those inferences can be. For example, if you can observe and interact with a student as he or she solves the tasks, you can be more confident in your interpretation.

However, even the static, written assessments from which we drew the examples provided here allowed us to assess these students' understandings to some degree.

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